A simple Cartesian treatment of planetary motion

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Abstract. Two famous theorems are proved here in a simple manner. First, it is proved that planets pursuing Keplerian trajectories have accelerations which conform to Newton's central $1/R^2$ equation. Then it is proved that, conversely, planetary orbits must be Keplerian if Newton's central $1/R^2$ equation holds true.

Zusammenfassung. Zwei berühmte Theoreme werden in diesem paper mit einer einfachen Methode bestätigt. Erstens, es wird gezeigt, daß Planeten, die Kepler-Bahnen folgen in einer Weise beschleunigt werden, die mit Newtons zentraler $1/R^2$ Gleichung überein stimmen. Dann wird gezeigt, daß diese Behauptung symmetrisch ist. Falls Newtons zentrale $1/R^2$ Gleichung wahr ist, dann muß die Planetenbahn den Kepler-Gesetzen folgen.

1. Introduction

Sir Isaac Newton codiscovered calculus and established the three laws of motion which bear his name. He is also responsible for the inverse-square law which is more accurate than any prior law of gravity. Perhaps Newton's greatest achievement was to prove that his inverse-square law is consistent with the older laws of Johannes Kepler.

Supposing that planets move according to the three laws discovered by Kepler, it then follows that planets' accelerations are given by Newton's central inverse-square equation (which is equation (12) below). This historic theorem can be proved using only basic calculus, and it is also easy to prove the converse theorem according to which Newton's equation implies Kepler's laws. Both of these famous theorems are proved here in a straightforward manner, using Cartesian coordinates throughout. There is no need for the usual transformations from Cartesian coordinates to polar coordinates and back again.

The two theorems which are proved below were first published in Newton's 1687 *Philosophiae Naturalis Principia Mathematica*. Not only is that book purposely 'abstruse' (Christianson 1984, p 290) but some people even question whether its proofs are entirely legitimate (see Arnol'd 1990). The proofs below are much less daunting.

After reviewing Kepler's laws in section 2, I prove in section 3 that these laws necessarily imply the central $1/R^2$ equation. Then Kepler's laws are recovered from the $1/R^2$ equation in section 4, but without need of the 'clever tricks' that are often used when polar coordinates are employed (Temple and Tracy 1992). A few authors have used methods similar to that of section 4 in order to recover Kepler's first law (Hart 1880, Wintner 1941, Abraham and Marsden 1978), but all of those proofs include a calculation which is 'totally lacking of any perceptible motivation' (Weinstock 1991). The calculations below have the advantage of being well-motivated, in that each step in section 4 follows naturally from what precedes it. By the way, I will assume that motion is confined to a plane, although this simplifying assumption is easily justified (Smart 1977).

2. Review of Kepler's laws

Kepler deduced his laws from empirical data supplied by the astronomer Tycho Brahe. Kepler's laws are:

I. Each planet moves along an ellipse with the Sun at a focus.

II. The line between a planet and the Sun sweeps out equal areas in equal times.

III. The square of a revolution's duration, divided by the cube of the orbit's greatest width, is the same for all planets.

Kepler introduced the first two laws in his 1609 *Astronomia Nova*. The third or 'harmonic' law was

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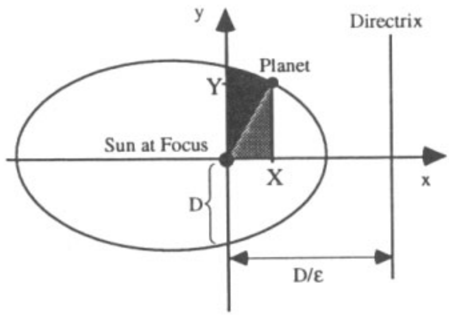

Figure 1. Diagram of Keplerian motion.

suggested in his 1619 *Harmonice Mundi* and is often stated in terms of the length 'a' of the semimajor axis ('a' is half the orbit's greatest width). The discovery of Kepler's laws was the greatest advance since Aristarchus deduced nineteen centuries earlier that planets circle the Sun (see Heath 1981).

Recall that ellipses are the closed curves formed by intersecting a circular cone and a plane. The ancient Greeks proved (see Heath 1981) that, everywhere along an ellipse, the distance to a point (the 'focus') divided by the distance to a line (the 'directrix') is a constant 'eccentricity' ϵ. A beautiful proof of this focus-directrix property was devised in 1822 by G P Dandelin, for both open ($\epsilon \geq 1$) and closed ($0 \leq \epsilon < 1$) conic sections (see Shenk 1977 or Thomas and Finney 1984).

Kepler's laws can be translated into equations by considering a planet as a point-particle in the x–y plane, having coordinates (X, Y) at time t (see figure 1). The Sun is at the origin and the planet's directrix is perpendicular to the x-axis at a distance D/ϵ from the Sun. D is called the 'semi-latus-rectum' of the conic section (measured values of ϵ and D are given in table 1). According to Kepler's First Law, the distance $R \equiv \sqrt{X^2 + Y^2}$ from the planet to the Sun is given by:

$$R = D - \epsilon X. \tag{1}$$

Kepler's Second Law can be formulated in similarly simple terms. If the planet crosses the y-axis at time t_0, the area swept between t_0 and t equals the area under the curve minus the triangular area beneath the line from Sun to planet (see figure 1). Hence,

$$\int_0^X y\,\mathrm{d}x - XY/2 = C(t - t_0) \tag{2}$$

where C is the constant ratio of area swept to time elapsed.

The orbit's total area divided by a revolution's duration (i.e. divided by its 'period') is clearly given by C. Also, it is not difficult to prove that the area of an ellipse is πab with the semimajor and semiminor axes given by $a = D[1 - \epsilon^2]^{-1}$ and $b = D[1 - \epsilon^2]^{-1/2}$ respectively (these two formulae can be easily derived using equation (1)). Consequently,

Table 1. Shapes and sizes of orbits.

Name	ϵ	D (m)
Mercury	0.21	5.5×10^{10}
Venus	0.01	1.1×10^{11}
Earth	0.02	1.5×10^{11}
Halley's Comet	0.97	1.6×10^{11}
Mars	0.09	2.3×10^{11}
Ceres	0.08	4.1×10^{11}
Jupiter	0.05	7.8×10^{11}
Saturn	0.06	1.4×10^{12}
Uranus	0.05	2.9×10^{12}
Neptune	0.01	4.5×10^{12}
Pluto	0.25	5.5×10^{12}

Kepler's Third Law is:

$$C^2/D = K \tag{3}$$

where the constant K is the same for all planets (about 3.3×10^{19} in MKS units). In summary, Kepler's laws are (1), (2), and (3).

3. Proof of the central inverse-square equation

The acceleration of planets will now be calculated by differentiating Kepler's laws. Differentiating (1) and (2) with respect to time t yields:

$$\frac{1}{R}\left(X\frac{\mathrm{d}X}{\mathrm{d}t} + Y\frac{\mathrm{d}Y}{\mathrm{d}t}\right) = -\epsilon\frac{\mathrm{d}X}{\mathrm{d}t} \tag{4}$$

and

$$Y\frac{\mathrm{d}X}{\mathrm{d}t} - X\frac{\mathrm{d}Y}{\mathrm{d}t} = 2C \tag{5}$$

respectively†. Some algebra applied to equations (4), (5), and (1) makes it clear that:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{2C}{D}\frac{Y}{R} \tag{6}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -\frac{2C}{D}\frac{X}{R} - \frac{2C\epsilon}{D}. \tag{7}$$

In order to calculate the two acceleration components, it is easier to differentiate (5) and (6) than (6) and (7). From (5) it follows immediately that:

$$Y\frac{\mathrm{d}^2X}{\mathrm{d}t^2} - X\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} = 0. \tag{8}$$

Differentiation of the right-hand side of (6) is facilitated by the following identity which is based solely upon the definition of R:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Y}{R}\right) = \frac{X}{R^3}\left(X\frac{\mathrm{d}Y}{\mathrm{d}t} - Y\frac{\mathrm{d}X}{\mathrm{d}t}\right). \tag{9}$$

† The more knowledgeable reader will notice that (5) expresses 'angular momentum conservation'. This equation is mathematically equivalent to the area law (2), and also to the 'central force' equation (8).

Thus, by differentiating (6) and including (5) and (3), one gets:

$$\frac{d^2X}{dt^2} = \frac{-4KX}{R^3}. \tag{10}$$

By (8) and (10),

$$\frac{d^2Y}{dt^2} = \frac{-4KY}{R^3}. \tag{11}$$

Equations (10) and (11) can be written compactly in terms of vectors.

$$\frac{d^2\mathbf{R}}{dt^2} = \frac{-4K\mathbf{R}}{R^3}. \tag{12}$$

This is Newton's central $1/R^2$ equation. Equation (12) expresses Newton's law of gravity for the special situation where planetary mass is negligibly small (the constant K is proportional to the Solar mass).

4. Recovery of Kepler's laws

It remains to be seen whether a bounded orbit could possibly satisfy (12) if it not Keplerian. In other words, could a planet be accelerating according to (12) and yet violate Kepler's laws? It will now be proved that such an orbit is impossible, by recovering Kepler's laws from (12). Equations (10) and (11) lead to (8), and integrating (8) retrieves (5) and (2). Putting (5) into the useful identity† (9) gives:

$$\frac{d}{dt}\left(\frac{Y}{R}\right) = \frac{-2CX}{R^3}. \tag{13}$$

So, by (10),

$$\frac{Y}{R} = \frac{C}{2K}\frac{dX}{dt} + A \tag{14}$$

where A is a constant of integration. Interchanging X and Y in (9) produces another identity which together with (5) yields

$$\frac{d}{dt}\left(\frac{X}{R}\right) = \frac{2CY}{R^3}. \tag{15}$$

So, by (11),

$$\frac{X}{R} = \frac{-C}{2K}\frac{dY}{dt} + B \tag{16}$$

where B is another constant of integration. Plugging (14) and (16) into (5) yields:

$$(C^2/K) + AY + BX = R. \tag{17}$$

If $A = B = 0$, this describes a circle. If not, (17) represents a conic section with focus at the origin, eccentricity $[A^2 + B^2]^{1/2}$, and directrix given by:

$$(C^2/K) + Ay + Bx = 0. \tag{18}$$

This interpretation of (17) follows from a simple rule of analytic geometry: the distance from a point (x_0, y_0) to a line $ax + by + c = 0$ is given by $|ax_0 + by_0 + c|[a^2 + b^2]^{-1/2}$. This rule is discussed by Shenk (1977) and by Thomas and Finney (1984). When applied to (18), this same rule requires that the focus–directrix distance is as described by (3). Consequently, if Newton's central inverse-square equation holds true then all bounded orbits must satisfy Kepler's laws, which was to be demonstrated.

† This vital identity arises naturally in the context of section 3. However, when this context is absent, the identity is 'pulled out of the air and the only justification seems to be that it works' (Peters 1991).

5. Conclusion

The task of demonstrating the relationship between the laws of Kepler and Newton was 'the major scientific problem of the [seventeenth] century' (Cohen 1982). The simple technique presented in this article may enable more people to appreciate this relationship.

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