Relativistic charged-particle motion in a constant field according to the Lorentz force law

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A new and simple method is presented for integrating the equations of motion of a classical relativistic particle subject to a Lorentz force, given electromagnetic fields that are constant in space-time. This integration method leads naturally to a manifestly covariant form of solution.

I. BACKGROUND

Imagine that a particle is in an external electromagnetic field which is uniform in space and static in time. The particle is a nice comprehensible classical particle, moving in obedience to the Lorentz force law. The particle's initial four-velocity is V^μ and its initial position is P^μ.

Physicists, who essentially are in the business of foretelling the future, have only one basic question about this particle: Where will it be later on? For those who would like to skip right to the bottom line, the answer is given by Eq. (31) below. Usually, this type of problem is solved assuming that the constant fields or the initial conditions satisfy certain constraints, but the approach here is to assume nothing of this kind.

Let's begin by taking a quick look at the case of a free particle. It is no great revelation that the following equations can be easily solved:

$$\frac{d^2x^\nu}{d\tau^2}=0 \quad \text{and} \quad \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}=-1. \tag{1}$$

The covariant expression for the four-velocity is

$$\frac{dx^\mu}{d\tau}=V^\mu, \tag{2}$$

with

$$V^\mu V_\mu=-1, \tag{3}$$

where V^μ is constant. Integrating (2) leads to the general solution of (1), where P^μ is another constant four-vector:

$$x^\mu=V^\mu\tau+P^\mu. \tag{4}$$

This is the solution for a free classical particle. Fortunately, life is more complicated and interesting than this.

Consider now a relativistic particle that is not free, but instead is immersed in a constant external electromagnetic field that exerts a Lorentz force on the particle. Leaving aside radiation reaction, the well-known equations of motion are

$$\frac{d^2x^\mu}{d\tau^2}=\frac{q}{m}F^\mu{}_\sigma\frac{dx^\sigma}{d\tau} \quad \text{and} \quad \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}=-1. \tag{5}$$

These equations can be solved exactly, and several integration methods have been employed for this purpose.[1–3] Now a simple new method will be used to solve (5). As far as I have been able to determine, the solution of (5) has never before been published in a tensor form analogous to (4); this apparent gap in the literature will be filled using the new method of integration presented here.

Note that constant electromagnetic fields satisfy Maxwell's equations in vacuum. Those equations are

$$F^{\mu\nu}{}_{,\mu}=0 \quad \text{and} \quad G^{\mu\nu}{}_{,\mu}=0, \tag{6}$$

with

$$G_{\alpha\beta}\equiv\tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}. \tag{7}$$

Clearly, if $F_{\alpha\beta}$ is uniform and static throughout a region of space-time then Maxwell's equations will be satisfied in that region.

With regard to notation, units are chosen here so that the speed of light is one. Greek indices run from 0 to 3. Repeated indices are summed. Commas denote partial differentiation. Greek indices are raised and lowered by summation with the Minkowski tensor $\eta^{\mu\nu}=\eta_{\mu\nu}$, which is diagonal and has components $(-1,1,1,1)$. The constant charge of a particle is "q" and its constant rest mass is "m." The proper time of the particle is "τ" although it will be handy to instead use the parameter "ζ," which is by definition "$q\tau/m$." The Levi-Civita tensor is defined as follows:

$$\epsilon^{\alpha\beta\mu\nu}\equiv\begin{cases}1, & \text{if } \alpha\beta\mu\nu \text{ is an even permutation of } 0\ 1\ 2\ 3,\\ -1, & \text{if } \alpha\beta\mu\nu \text{ is an odd permutation of } 0\ 1\ 2\ 3,\\ 0, & \text{otherwise.}\end{cases} \tag{8}$$

The electromagnetic field strength tensor $F^{\mu\nu}$ is antisymmetric and, by assumption, is constant.

II. SOLUTION OF THE EQUATIONS OF MOTION

Changing independent variables in (5), and letting dots denote differentiation with respect to ζ, we have

$$\ddot{x}^\mu=F^\mu{}_\sigma\dot{x}^\sigma. \tag{9}$$

It will be convenient to write (9) in matrix notation:

$$\ddot{x}=F\dot{x}. \tag{10}$$

The square matrices F and G can be written as follows, in terms of the ordinary electric and magnetic field components:

$$F \equiv [F^{\mu}{}_{\sigma}] = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{11a}$$

and

$$G \equiv [G^{\mu}{}_{\sigma}] = \begin{bmatrix} 0 & B_x & B_y & B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix}. \tag{11b}$$

It is easy to find a power series solution of (10) for $\dot{x}$, by analogy with the series for the exponential function:

$$\dot{x} = \left[\sum_{n=0}^{\infty} \frac{1}{n!} (\zeta F)^n \right] \frac{mV}{q}. \tag{12}$$

The constant vector "V" is the initial four-velocity, and of course $(F)^0$ is by definition the identity matrix I. Although it is evident that (12) satisfies (10), it is not immediately clear how to write $\dot{x}$ in terms of elementary functions instead of as a series. This can be accomplished by noting that Eqs. (11) yield the following algebraic identities:

$$F^2 = G^2 + Ik_1 \tag{13}$$

and

$$FG = GF = \tfrac{1}{2} Ik_2, \tag{14}$$

so

$$F^3 = \tfrac{1}{2}k_2 G + k_1 F, \tag{15}$$

with

$$k_1 \equiv E^2 - B^2 = \tfrac{1}{2}\,\mathrm{Tr}(F^2) = -\tfrac{1}{2}F^{\sigma\tau}F_{\sigma\tau} \tag{16}$$

and

$$k_2 \equiv 2(\mathbf{E}\cdot\mathbf{B}) = \tfrac{1}{2}\,\mathrm{Tr}(FG) = \tfrac{1}{2}F^{\sigma\tau}G_{\tau\sigma}. \tag{17}$$

If we were to plug (15) and (14) into (12), then we would wind up with an equation of the following form:

$$\dot{x} = \frac{m}{q}\,[If_1(\zeta) + Ff_2(\zeta) + F^2 f_3(\zeta) + Gf_4(\zeta)]V, \tag{18}$$

where

$$f_1(0) = 1 \quad \text{and} \quad f_2(0) = f_3(0) = f_4(0) = 0. \tag{19}$$

Once these four scalar functions have been determined, then (18) can be easily integrated, and the analog of (4) discovered.

Plug (18) into (10), simplify with (14) and (15), and obtain four equations for the four scalar functions:

$$f_1 + f_3 k_1 - \dot{f}_2 = 0, \quad f_2 - \dot{f}_3 = 0,$$
$$f_4 k_2 - 2\dot{f}_1 = 0, \quad \text{and} \quad f_3 k_2 - 2\dot{f}_4 = 0. \tag{20}$$

It will be convenient to tentatively assume that k_2 is nonzero (the zero case will be dealt with later). Equations (20) thus dictate that

$$f_2 = \dot{f}_3, \quad f_1 = \ddot{f}_3 - f_3 k_1, \quad f_4 = \frac{2}{k_2}\,[\dddot{f}_3 - k_1 \dot{f}_3], \tag{21}$$

and

$$\ddddot{f}_2 - k_1 \ddot{f}_3 - \tfrac{1}{4}(k_2)^2 f_3 = 0. \tag{22}$$

Furthermore, Eqs. (19) give us four initial conditions:

$$f_3(0) = \dot{f}_3(0) = \dddot{f}_3(0) = 0 \quad \text{and} \quad \ddot{f}_3(0) = 1. \tag{23}$$

This entire problem has been boiled down to solving (22), which is a linear ordinary differential equation with constant coefficients. The characteristic equation of (22) is

$$\lambda^4 - k_1 \lambda^2 - \tfrac{1}{4}(k_2)^2 = 0. \tag{24}$$

Equation (24) is simply a quadratic equation in λ^2, so the four roots of this equation are $\pm a$ and $\pm ib$:

$$a \equiv \sqrt{\frac{\sqrt{(k_1)^2 + (k_2)^2} + k_1}{2}} \tag{25a}$$

and

$$b \equiv \sqrt{\frac{\sqrt{(k_1)^2 + (k_2)^2} - k_1}{2}}. \tag{25b}$$

Notice that a and b are both real and positive.[4] Now we can write down the solution of (22):

$$f_3 = C_1 \exp(a\zeta) + C_2 \exp(-a\zeta) + C_3 \cos(b\zeta) + C_4 \sin(b\zeta). \tag{26}$$

Equations (23) require that the four undefined constants in (26) take certain values:

$$C_4 = 0, \quad C_1 = C_2 = \frac{1}{2(a^2 + b^2)}, \quad C_3 = \frac{-1}{a^2 + b^2}. \tag{27}$$

Equations (27) can be inserted into Eqs. (26), which, in turn, can be inserted into (21), all of which gets us to the following four equations:

$$f_3 = \frac{1}{a^2 + b^2}\,[\cosh(a\zeta) - \cos(b\zeta)], \tag{28a}$$

$$f_2 = \frac{1}{a^2 + b^2}\,[a \sinh(a\zeta) + b \sin(b\zeta)], \tag{28b}$$

$$f_1 = \frac{1}{a^2 + b^2}\,[b^2 \cosh(a\zeta) + a^2 \cos(b\zeta)], \tag{28c}$$

$$f_4 = \frac{2}{k_2(a^2 + b^2)}\,[ab^2 \sinh(a\zeta) - ba^2 \sin(b\zeta)]. \tag{28d}$$

Equations (28) and (18) point to this result:

$$\dot{x} = \frac{m}{q(a^2 + b^2)} \Big[I(b^2 \cosh a\zeta + a^2 \cos b\zeta) + F(a \sinh a\zeta + b \sin b\zeta) + F^2(\cosh a\zeta - \cos b\zeta) + G\,\frac{k_2}{|k_2|}\,(b \sinh a\zeta - a \sin b\zeta) \Big] V. \tag{29}$$

Equation (29) can be easily integrated to give

$$x=\frac{m}{q(a^2+b^2)}\left[I\left(\frac{b^2\ \sinh a\zeta}{a}+\frac{a^2\ \sin b\zeta}{b}\right)\right.$$
$$+F(\cosh a\zeta-\cos b\zeta)+F^2\left(\frac{\sinh a\zeta}{a}-\frac{\sin b\zeta}{b}\right)$$
$$\left.-G\frac{k_2}{|k_2|}\left(\frac{b-b\ \cosh a\zeta}{a}+\frac{a-a\ \cos b\zeta}{b}\right)\right]V+P. \tag{30}$$

It is straightforward to convert (30) from a matrix format to a tensor format:

$$x^\mu=\frac{m}{q(a^2+b^2)}\left[\eta^\mu{}_\lambda\left(\frac{b^2\ \sinh a\zeta}{a}+\frac{a^2\ \sin b\zeta}{b}\right)\right.$$
$$+F^\mu{}_\lambda(\cosh a\zeta-\cos b\zeta)+F^\mu{}_\sigma F^\sigma{}_\lambda\left(\frac{\sinh a\zeta}{a}\right.$$
$$\left.-\frac{\sin b\zeta}{b}\right)-G^\mu{}_\lambda\frac{k_2}{|k_2|}\left(\frac{b-b\ \cosh a\zeta}{a}\right.$$
$$\left.\left.+\frac{a-a\ \cos b\zeta}{b}\right)\right]V^\lambda+P^\mu. \tag{31}$$

The general solution of (5) is (31) together with the definitions (25), (17), and (16). The constant V^μ is the four-velocity when $\zeta=0$, and the constant P^μ is the position in space-time when $\zeta=0$.

III. THE SINGULAR CASE

The solution (31) is perfectly well behaved for k_2 not equal to zero, but, if the constant k_2 vanishes, then ''a'' and/or ''b'' vanishes. This situation, in which the dot product of the electric and magnetic fields is equal to zero, can be dealt with by applying L'Hôpital's rule to (31)—or (more rigorously) by going back to Eq. (20). Using the L'Hôpital method, it is easy to rattle off the results. The limit of (31) as ''a'' approaches zero is as follows:

$$x^\mu=\frac{m}{qb^2}\left[\eta^\mu{}_\lambda(b^2\zeta)+F^\mu{}_\lambda(1-\cos b\zeta)\right.$$
$$\left.+F^\mu{}_\sigma F^\sigma{}_\lambda\left(\zeta-\frac{\sin b\zeta}{b}\right)\right]V^\lambda+P^\mu. \tag{32}$$

This is the limit of (31) as ''b'' approaches zero:

$$x^\mu=\frac{m}{qa^2}\left[\eta^\mu{}_\lambda(a^2\zeta)-F^\mu{}_\lambda(1-\cosh a\zeta)\right.$$
$$\left.-F^\mu{}_\sigma F^\sigma{}_\lambda\left(\zeta-\frac{\sinh a\zeta}{a}\right)\right]V^\lambda+P^\mu. \tag{33}$$

The limit of (32) as ''b'' approaches zero, or of (33) as ''a'' approaches zero, is even shorter:

$$x^\mu=\frac{m}{q}\left[\eta^\mu{}_\lambda(\zeta)-F^\mu{}_\lambda\left(\frac{\zeta^2}{2}\right)-F^\mu{}_\sigma F^\sigma{}_\lambda\left(\frac{\zeta^3}{6}\right)\right]V^\lambda+P^\mu. \tag{34}$$

If the electromagnetic field vanishes in (34), then we recover (4), as we ought. In summary, the solution for $k_2=0$ and $k_1<0$ is (32). The solution for $k_2=0$ and $k_1>0$ is (33). The solution for $k_2=0$ and $k_1=0$ is (34). All of these solutions, as well as (31), can be written in three-dimensional notation whenever it is convenient to do so, and this technique is illustrated in the Appendix, with respect to (32).

IV. CONCLUSION

A new and comparatively simple method has been used to solve the Lorentz force law for a relativistic particle immersed in constant fields. The entire derivation has been manifestly covariant, and no assumptions have been made about the initial conditions or about the constant fields. The final results are manifestly covariant, concise, and comprehensive.

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APPENDIX

The case of Eq. (32) will now be briefly discussed, in order to illustrate how the results of this article can be analyzed using three-dimensional notation. Recall that, since k_2 vanishes and k_1 is negative, the electric and magnetic fields are perpendicular and $b=\sqrt{B^2-E^2}$. If we choose coordinates so the particle starts at rest from the origin, then (32) and (11) say that the trajectory is described as follows:

$$\mathbf{r}=\frac{m}{qb^2}\left[\mathbf{E}(1-\cos b\zeta)+(\mathbf{E}\times\mathbf{B})\left(\zeta-\frac{\sin b\zeta}{b}\right)\right], \tag{A1}$$

$$t=\frac{m\zeta}{q}+\frac{m}{qb^3}E^2[b\zeta-\sin b\zeta]. \tag{A2}$$

Clearly, the dot product $\mathbf{r}\cdot\mathbf{B}$ vanishes, so motion is in a plane perpendicular to the magnetic field. It is convenient to choose coordinates so that the magnetic field points in the direction of the positive z axis, which means motion is in the x-y plane. Let the electric field lie in the direction of the positive x axis. Equation (A1) then breaks down into a more tractable form:

$$z=0, \tag{A3}$$

$$x=\frac{-mE}{qb^2}[\cos(b\zeta)-1], \tag{A4}$$

$$y=\frac{mBE}{qb^3}[\sin(b\zeta)-b\zeta]. \tag{A5}$$

We get where we want to go by algebraically eliminating the parameter ζ and the trigonometric functions from (A2), (A4), and (A5):

$$\frac{[x-mE/qb^2]^2}{[mE/qb^2]^2}+\frac{[y+Et/B]^2}{[mE/qbB]^2}=1. \tag{A6}$$

At any fixed time ''t'' this equation represents an ellipse. Thus the particle moves along an ellipse in a plane perpendicular to the magnetic field, and the ellipse itself moves sideways in a direction perpendicular to both the electric field and to the magnetic field. This ellipse moves with speed

E/B, it has eccentricity $\epsilon = E/B$, and it has semi-latus-rectum $D = mE \div |q|B^2$.

If the particle moves at a nonrelativistic speed, then E/B must be a very small fraction (recall that the speed of light is unity). Consequently, $b \cong B$, and hence in the nonrelativistic case equations (A4) and (A5) reduce to the equations for a cycloid.

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[2]Jean-François Dumais, ''Eigenvectors of the electromagnetic field tensor and relativistic charged particle motions,'' Am. J. Phys. **53**(3), 264–266 (1985).

[3]Nikos Salingaros, ''Particle in an External Electromagnetic Field. II. The Exact Velocity in a Constant and Uniform Field,'' Phys. Rev. D **31**(12), 3150–3156 (1985). Salingaros compares various methods of solving this problem, and points out that two published methods (of D. Hestenes and A. H. Taub) contain errors.

[4]The treatment of Salingaros (see Ref. 1 above) is less general than the present treatment, inasmuch as Table I of his article gives the dot product of the electric and magnetic fields as the product of two non-negative square roots, and so the dot product must itself be non-negative.