A parameter-independent formulation of the retarded electromagnetic fields

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A ''parameter-independent'' formulation of the retarded fields for a point charge ties together the various formulations that have appeared elsewhere during the last 100 years. The parameter-independent formulation clarifies and simplifies the relationship between those other formulations. Sometimes it is useful to formulate the retarded fields using the ordinary retarded coordinate time as a parameter of motion,[1] sometimes it is more useful to use the retarded proper time as a parameter,[2] and sometimes it can be convenient to use the nonretarded coordinate time as the parameter of motion.[3] These three formulations are special cases of the parameter-independent formulation.

Regardless of how one formulates the retarded fields, their physical properties remain the same. Of course, they satisfy the Maxwell equations in free space, energy-momentum is conserved in free space, etc.

Feynman discussed the retarded fields in his lecture series.[3] He later commented that the retarded field formulae, together with the Lorentz force law, incorporate and account for virtually all aspects of classical electrodynamics: ''[b]arring quantum mechanics and self action problems (both of which I am explicit about) there is only the modification in general relativity if spacetime is curved.''[4]

Here now is a ''parameter-independent'' formulation of the retarded fields of a point charge:

$$F_{\mu\nu}=\frac{q(R_\nu v_\mu - R_\mu v_\nu)}{r^3}[R^\sigma a_\sigma - v^\beta v_\beta] + \frac{q(R_\nu a_\mu - R_\mu a_\nu)}{r^2}. \tag{1}$$

The retardation condition is $R^\alpha R_\alpha = 0$ with $R_\alpha \equiv x_\alpha - z_\alpha(p)$. The constant charge of the particle is denoted by ''q.'' The field coordinates are represented by x^α and the coordinates of the source particle by $z^\alpha(p)$. Repeated indices are summed, Greek indices run from zero to three (Latin indices will run from one to three), and Greek indices are raised and lowered with the diagonal Minkowski tensor $\eta^{\mu\nu}=\eta_{\mu\nu}$ which has elements $(-1,1,1,1)$. $a^\beta \equiv \dot{v}^\beta$ and $v^\beta \equiv \dot{z}^\beta$ with dots denoting ordinary differentiation with respect to the parameter ''p;'' $r \equiv -R^\sigma v_\sigma$. Equation (1) is written in a parameter-independent way,[5] so we could either use proper time as a parameter, or coordinate time, or any other parameter of motion. Units are chosen so that the speed of light is unity.

The parameter independence of (1) is very easy to prove. Suppose we introduce a new parameter p'. We then have the following transformation:

$$v_\mu = \frac{dp'}{dp} v'_\mu, \quad a_\mu = \left[\frac{dp'}{dp}\right]^2 a'_\mu + \frac{d^2p'}{dp^2} v'_\mu, \quad r = \frac{dp'}{dp} r'. \tag{2}$$

If we plug these formulae back into Eq. (1) and remove the primes, then we wind up with the very same equation we started out with.

In order to recover the formulation that uses retarded proper time as the parameter of motion,[2] we need only set $v^\sigma v_\sigma = -1$ in Eq. (1). That's all there is to it.

To recover the formulations that use coordinate time as the parameter of motion, keep in mind that the electric and magnetic fields are defined like this:

$$E_n \equiv F_{n0}, \quad B_1 \equiv F_{23}, \quad B_2 \equiv F_{31}, \quad B_3 \equiv F_{12}. \tag{3}$$

According to Eq. (1), we have

$$R_\mu F_{\alpha\nu} = R_\alpha F_{\mu\nu} - R_\nu F_{\mu\alpha}. \tag{4}$$

Therefore, setting μ equal to zero, α equal to ''n,'' and ν equal to ''d,'' we get

$$F_{nd} = \frac{R_n F_{d0} - R_d F_{n0}}{R^0}. \tag{5}$$

And, using Eqs. (3), we recover the usual expression for the magnetic field:

$$\mathbf{B} = \frac{\mathbf{R}\times\mathbf{E}}{R}. \tag{6}$$

Here we have set $R \equiv \sqrt{R_n R_n}$.

Now, let us recover from Eq. (1) the formula for the electric field. We immediately get

$$E_n = \frac{q(R_0 v_n - R_n v_0)}{r^3}[R^\sigma a_\sigma - v^\beta v_\beta] + \frac{q(R_0 a_n - R_n a_0)}{r^2}. \tag{7}$$

If we let retarded coordinate time act as the parameter of motion, then the electric field takes this form:

$$E_n = \frac{q(R_n - R v_n)}{(R - R_d v_d)^3}[R_b a_b + 1 - v^2] - \frac{q R a_n}{(R - R_d v_d)^2}. \tag{8}$$

Switching to vector notation, this gives the well-known result:[1]

$$\mathbf{E} = \frac{q(\mathbf{R} - R\mathbf{v})}{(R - \mathbf{R}\cdot\mathbf{v})^3}[\mathbf{R}\cdot\mathbf{a} + 1 - v^2] - \frac{qR\mathbf{a}}{(R - \mathbf{R}\cdot\mathbf{v})^2}. \tag{9}$$

It remains to recover from Eq. (7) the formula for the electric field such that the parameter of motion is the nonretarded coordinate time at a field point fixed in space. Differentiating the retardation condition $x^0 - z^0 = R$ with respect to the parameter $t \equiv x^0$ we get

$$v^0 = 1 - \frac{dR}{dt}, \tag{10a}$$

where

$$\frac{dR}{dt} = \frac{-R_d v_d}{R}. \tag{10b}$$

Thus

$$r = R, \quad a^0 = -\frac{d^2R}{dt^2}, \quad \frac{d}{dt}\left[R\frac{dR}{dt} + R_d v_d\right] = 0. \tag{11}$$

Consequently, Eq. (7) gives

$$\begin{aligned} E_n &= \frac{q(R_0 v_n - R_n v_0)}{R^3}\left[1 - 2\frac{dR}{dt}\right] + \frac{q(R_0 a_n - R_n a_0)}{R^2} \\ &= \frac{qR_n}{R^3} + \frac{qR(dR_n/dt) - 3qR_n(dR/dt)}{R^3} \\ &\quad - \frac{2q(dR_n/dt)(dR/dt) - qR(d^2R_n/dt^2) + qR_n(d^2R/dt^2) - 2q(R_n/R)(dR/dt)^2}{R^2}. \end{aligned} \tag{12}$$

It is fairly obvious from (12) that the electric field can finally be expressed by the Heaviside–Feynman formula:[3]

$$E_n=\frac{qR_n}{R^3}+qR\,\frac{d}{dt}\left[\frac{R_n}{R^3}\right]+q\,\frac{d^2}{dt^2}\left[\frac{R_n}{R}\right]. \qquad (13)$$

We have thus succeeded in easily deriving the three usual formulations of the retarded fields as special cases of the parameter-independent formulation. This procedure can be contrasted with treatments in which one of the usual formulations is derived directly from another.[6]

ACKNOWLEDGMENT

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[1] J. Reitz and F. Milford, *Foundations of Electromagnetic Theory* (Addison–Wesley, Reading, MA, 1967), p. 406. This formula was first discovered by Alfred Marie Liénard 100 years ago, in 1898. See A. O'Rahilly, *Electromagnetic Theory* (Dover, New York, 1965), Vol 1, pp. 217–218.
[2] W. Pauli, *Theory of Relativity* (Pergamon, New York, 1958), p. 91.
[3] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison–Wesley, Reading, MA, 1964), p. I-28-2. This formulation was first discovered by Oliver Heaviside in 1902, and rediscovered by Feynman in 1950. See J. J. Monoghan, ''The Heaviside–Feynman expression for the fields of an accelerated dipole,'' J. Phys. A **1**, 112–117 (1968).
[4] R. P. Feynman (personal communication, February 10, 1981).
[5] The Lorentz force law can likewise be expressed in a parameter-independent way. See A. T. Hyman, ''Charged-Particle Motion in a Constant Field,'' Am. J. Phys. **65** (8), 688 (1997).
[6] D. Griffiths and M. Heald, ''Time-Dependent Generalizations of the Biot–Savart and Coulomb Laws,'' Am. J. Phys. **59** (2), 111 (1991). Also see Alan M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978), Appendix H.

Erratum: ''Resource Letter GPP-1: Geometric phases in physics'' [Am. J. Phys. 65 (3), 178–185 (1997)]

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After this resource letter was written the following interesting papers on the geometric phase were brought to our attention:

[143] E. Gozzi and W. D. Thacker, ''Classical adiabatic holonomy in a Grassmannian system,'' Phys. Rev. D **35** (8), 2388–2397 (1987). (A)
[144] R. Bhandari, ''SU(2) phase jumps and geometric phases,'' Phys. Lett. A **157**, 221–225 (1991). (I)
[145] A. Mostafazadeh and A. Bohm, ''Topological Aspects of the Non-Adiabatic Berry Phase,'' J. Phys. A **26**, 5473–5479 (1993). (A)
[146] A. Bohm and A. Mostafazadeh, ''The Relation between the Berry and the Anandan–Aharonov Connections for U(N) bundles,'' J. Math. Phys. **35**, 1463–1470 (1994). (A)
[147] A. Mostafazadeh, ''Geometric Phase, Bundle Classification, and Group Representation,'' J. Math. Phys. **37**, 1218–1233 (1996). (A)
[148] R. Bhandari, ''Polarization of light and topological phases,'' Phys. Rep. **281**, 2–64 (1997). (I)

Another useful Keplerian average distance: The harmonic mean

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In two recent publications in this journal, Bucher and co-workers[1,2] derive interesting and useful expressions for planet–sun distances and positions, averaged over time, path, and angle. To do this, they examine relationships among various average velocities and distances, based on the construction of an ellipse and a confocal circle whose radius R is equal to a, the semimajor axis of the ellipse. By ''Kepler-average distance,'' Bucher and co-workers[1,2] mean a distance, which they denote by R, such that Kepler's third law holds in comparing the periods and orbits of two or more planets: $R_1^3/T_1^2=R_2^3/T_2^2$. They show that only the ''space-average distance'' (viz., the average over path),[1] which they denote as $\langle r\rangle_s$, is equal to R. In addition, however, they note that the Keplerian orbit ''time-average distance'' $\langle r\rangle_t$, while ''not...of much relevance in gravitational Kepler motion,''[2] finds utility in the Bohr–Sommerfeld (BS) model of the hy-

